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RADIATIVE TRANSFER IN AN ABSORBING–SCATTERING SLAB BOUNDED BY EMITTING AND REFLECTING SURFACES*

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Abstract – An exact integral theory of the planar radiative transfer with isotropic scattering and general boundary conditions is presented in the paper. The analytical solution to the problem is numerically processed for two different specializations of the emissivity and reflectivity properties of the bounding surfaces. Results are given for the total and angular radiation intensities as well as for the net radiative flux.

NOMENCLATURE

a, optical half-thickness;
c, albedo;
E_n, nth exponential integral;
I, angular radiation intensity;
I_0, total radiation intensity;
Q, source term;
q, net radiative flux;
q^+, q^-, forward, backward radiative flux.

Greek symbols

α, emitted power;
ε, emissivity of the boundary;
μ, cosine of the angle between the direction of radiation intensity and the positive τ axis;
$\rho = \rho^s + \rho^d$, boundary reflectivity;
$\rho^d = \gamma$, diffuse component of ρ;
$\rho^s = \beta$, specular component of ρ;
τ, optical coordinate.

Subscript

$i = 1, 2$, relative to the boundary $\tau = -a, \tau = a$.

INTRODUCTION

THE PAPER deals with the radiative transfer in an absorbing, scattering, emitting gray medium bounded by two parallel plane surfaces. The optical depth of the slab is $2a$ and the two bounding surfaces, which are kept at uniform temperature T_1 and T_2, respectively, are assumed to emit diffusely with emissivities ε_1 and ε_2, and to reflect both diffusely and specularly with reflectivities $\rho_i = \rho_i^d + \rho_i^s$ $(i = 1, 2)$.

The aim of the paper is to point out how the radiative transfer within the participating medium is affected by:

(i) the emissivity and reflectivity properties of the bounding surfaces as well as by the interaction between the bounding surfaces themselves;
(ii) the scattering properties of the participating medium.

For the case of constant transparent boundaries ($\varepsilon_i = \rho_i = 0$) with zero externally applied radiation, rigorous numerical results have been obtained by the integral transform method for both radiative transfer [1] and neutron transport [2], with isotropic and anisotropic scattering, respectively. The case ε_i, $\rho_i \neq 0$ has been extensively treated for isotropic scattering. To the authors' knowledge the most rigorous numerical results so far available in the literature are the ones, based on Case's eigenfunction expansion method [3], given by Lii and Ozisik [4] for the radiative flux through the boundary surfaces.

For the case of linearly anisotropic scattering approximate results have been given by Beach *et al.* [5] who resort to Case's method. Other approximate treatments of the anisotropic scattering case can be found in the literature, while an extension of the rigorous procedure adopted in [2] is now in progress.

In this paper a rigorous treatment of the isotropic case with the most general properties of the bounding surfaces is considered. The azimuthal symmetry of the internal source and of the boundary conditions is assumed throughout the discussion.

The paper consists of two sections. In Section 1 we present the exact theory for the problem under consideration by resorting to the linear Boltzmann equation for radiative transfer. An analytical series solution to the problem is obtained, the convergence rate of which can be estimated *a priori* to be very good. This analytical solution is then numerically processed in Section 2 for two different specializations of the properties of the bounding surfaces. Both angular and total radiation intensities are evaluated as well as the net radiative flux.

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1. THEORY

1.1. *Equations for the angular and total radiation intensities*

The physical situation sketched in the introduction can be adequately described by the linear integro-differential Boltzmann equation, which in the same notations as in [6] reads as

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau}+I(\tau,\mu)=Q(\tau,\mu)+\frac{c}{2}\int_{-1}^{1}I(\tau,\mu')\mathrm{d}\mu' \quad (-a\leqslant\tau\leqslant a;\quad -1\leqslant\mu\leqslant 1), \quad (1)$$

where $I(\tau,\mu)$ is the unknown angular radiation intensity. Equation (1) will be integrated over the following general boundary conditions, $\mu\in(0,1)$,

$$I(-a,\mu)=\frac{\alpha_1}{2\pi}h_1(\mu)+\beta_1 I(-a,-\mu)+\gamma_1 g_1(\mu)\int_0^1\mu' I(-a,-\mu')\mathrm{d}\mu',$$
$$I(a,-\mu)=\frac{\alpha_2}{2\pi}h_2(\mu)+\beta_2 I(a,\mu)+\gamma_2 g_2(\mu)\int_0^1\mu' I(a,\mu')\mathrm{d}\mu', \quad (2)$$

where

$$\alpha_i=\varepsilon_i\sigma T_i^4,\quad \beta_i=\rho_i^s,\quad \gamma_i=\rho_i^d,\quad (i=1,2) \quad (3a)$$

and $h_i(\mu)$ and $g_i(\mu)$ $(i=1,2)$, which are normalized to

$$\int_0^1\mu h_i(\mu)\mathrm{d}\mu=\int_0^1\mu g_i(\mu)\mathrm{d}\mu=1, \quad (3b)$$

account for the angular distributions of the emitted and the non-specularly reflected radiation (when $h_i(\mu)=g_i(\mu)=2$, then the boundary conditions of equations (2) coincide with those usually adopted in the literature [5]). Resorting to the Green's function method, it can be verified that the original integro-differential problem, equations (1) and (2), can be reformulated in a purely integral form, namely, $\mu\in(0,1)$,

$$\begin{aligned}I(\tau,\mu)=&\frac{c}{4\pi}\frac{1}{\mu}\int_{-a}^{\tau}\exp\left(-\frac{\tau-\tau'}{\mu}\right)I_0(\tau')\mathrm{d}\tau'\\&+\frac{1}{\mu}\int_{-a}^{\tau}\exp\left(-\frac{\tau-\tau'}{\mu}\right)Q(\tau',\mu)\mathrm{d}\tau'\\&+\frac{\alpha_1}{2\pi}h_1(\mu)\exp\left(-\frac{a+\tau}{\mu}\right)\\&+\beta_1\exp\left(-\frac{a+\tau}{\mu}\right)J_1(\mu)\\&+\gamma_1 g_1(\mu)\exp\left(-\frac{a+\tau}{\mu}\right)\int_0^1\mu' J_1(\mu')\mathrm{d}\mu',\\ I(\tau,-\mu)=&\frac{c}{4\pi}\frac{1}{\mu}\int_{\tau}^{a}\exp\left(\frac{\tau-\tau'}{\mu}\right)I_0(\tau')\mathrm{d}\tau'\\&+\frac{1}{\mu}\int_{\tau}^{a}\exp\left(\frac{\tau-\tau'}{\mu}\right)Q(\tau',\mu)\mathrm{d}\tau'+\frac{\alpha_2}{2\pi}h_2(\mu)\\&\times\exp\left(-\frac{a-\tau}{\mu}\right)+\beta_2\exp\left(-\frac{a-\tau}{\mu}\right)J_2(\mu)\\&+\gamma_2 g_2(\mu)\exp\left(-\frac{a-\tau}{\mu}\right)\int_0^1\mu' J_2(\mu')\mathrm{d}\mu',\end{aligned} \quad (4)$$

where

$$I_0(\tau)=2\pi\int_{-1}^{1}I(\tau,\mu)\mathrm{d}\mu \quad (5a)$$

is the total radiation intensity, and

$$J_1(\mu)=I(-a,-\mu),\quad J_2(\mu)=I(a,\mu). \quad (5b)$$

We now operate successively on equations (4) through two different steps. First by integrating over $\mu\in(0,1)$ and summing up the resulting equations we get a linear integral equation for $I_0(\tau)$ in terms of $J_1(\mu)$ and $J_2(\mu)$. Then, by specializing equations (4) at $\tau=\pm a$, we get two equations for $J_1(\mu)$ and $J_2(\mu)$, respectively, in terms of $I_0(\tau)$. Finally, by eliminating $J_1(\mu)$ and $J_2(\mu)$ from the resulting system, we get the linear integral equation for the total radiation intensity

$$I_0(\tau)=c\int_{-a}^{a}K(\tau,\tau')I_0(\tau')\mathrm{d}\tau'+S(\tau), \quad (6)$$

with kernel

$$K(\tau,\tau')=\frac{1}{2}E_1(|\tau-\tau'|)+\frac{1}{4\pi}\sum_{i=1}^{2}\int_0^1 H_i(\tau,\mu)\tilde{K}_i(\mu,\tau')\mathrm{d}\mu \quad (7a)$$

and known term

$$S(\tau)=S_0(\tau)+\sum_{i=1}^{2}\int_0^1 H_i(\tau,\mu)j_i(\mu)\mathrm{d}\mu. \quad (7b)$$

The functions $H_i(\tau,\mu)$, $\tilde{K}_i(\mu,\tau)$, $S_0(\tau)$ and $j_i(\mu)$ appearing in equations (7a) and (7b) are explicitly given in Appendix I.

Once equation (6) is solved for $I_0(\tau)$, from equations (4) there follows the angular radiation intensity $I(\tau,\mu)$, and then the net radiative flux

$$q(\tau)=q^+(\tau)-q^-(\tau)=\int_0^1\mu I(\tau,\mu)\mathrm{d}\mu-\int_0^1\mu I(\tau,-\mu)\mathrm{d}\mu \quad (8)$$

can also be evaluated.

1.2. *Solution of equation* (6)

In order to solve equation (6) we set

$$I_0(\tau)=\sum_{n=0}^{\infty}\left(\frac{2n+1}{2a}\right)^{1/2}\xi_n P_n\left(\frac{\tau}{a}\right), \quad (9)$$

where $P_n(x)$ denotes the nth Legendre polynomial. Introducing equation (9) in equation (6) we have

$$I_0(\tau)=c\sum_{n=0}^{\infty}\left(\frac{2n+1}{2a}\right)^{1/2}\xi_n\int_{-a}^{a}K(\tau,\tau')\times P_n\left(\frac{\tau'}{a}\right)\mathrm{d}\tau'+S(\tau). \quad (10)$$

As for the two solutions expressed by equation (9) and equation (10) respectively, we comment that, when they are truncated at the same term $n = N$, the solution, equation (10), turns out to be more accurate than the corresponding one of equation (9), especially for values of τ close to the boundaries [7].

According to classical methods, as illustrated for instance in [8], we now project equation (10) over the denumerable sequence of Legendre's polynomials $P_n(\tau/a)$. The unknown coefficients ξ_n are then recognized to be in turn solutions to the system of infinite linear algebraic equations

$$\xi_m = c\sum_{n=0}^{\infty} A_{mn}\xi_n + B_m, \quad (m = 0, 1, \ldots), \tag{11}$$

the matrix element A_{mn} being given by

$$A_{mn} = \frac{[(2m+1)(2n+1)]^{1/2}}{2a}\int_{-a}^{a} P_m\left(\frac{\tau}{a}\right)\mathrm{d}\tau \int_{-a}^{a} K(\tau,\tau')P_n\left(\frac{\tau'}{a}\right)\mathrm{d}\tau' \tag{12a}$$

and the known term by

$$B_m = \left(\frac{2m+1}{2a}\right)^{1/2}\int_{-a}^{a} S(\tau)P_m\left(\frac{\tau}{a}\right)\mathrm{d}\tau. \tag{12b}$$

By recalling the kernel and the source term of equation (6), one can evaluate A_{mn} and B_m, and then extract the ξ_n's from the system of equations (11). In practice, the infinite system of equations (11) is actually replaced by a finite system of order N ($N = 0, 1, 2, \ldots$). That the sequence of the approximate solutions of order N converges – in the norm of the Hilbert space $L_2(-a, a)$ – to the exact solution of equation (6) is guaranteed *a priori* as the kernel $K(\tau, \tau')$ is of finite double norm [7] in $L_2(-a, a)$ and the known term $S(\tau)$ also belongs to $L_2(-a, a)$, provided $1/c$ is not an eigenvalue of the linear integral operator generated by $K(\tau, \tau')$ [8]. The rate of convergence can be expected to be satisfactory as already proved by analogous applications of the theory in the allied field of neutron transport [10].

2. APPLICATIONS

2.1. *Generalities*

Two applications of the theory expounded in Section 1 have been considered, namely

(i) $Q = 0,\quad \beta_1 = \gamma_1 = 0,\quad h_1 = h_2 = g_2 = 2,$

that is, the bounding surface 1 is black. This case will be referred to as *the case of a black surface*.

(ii) $Q = 0,\quad \beta_1 = \beta_2 = 0,\quad h_1 = h_2 = g_1 = g_2 = 2,$

that is, both the bounding surfaces are not specularly reflecting. This case will be referred to as *the case of zero specular reflectivity*. For both these cases numerical results are obtained for the total and angular radiation intensities as well as for the net radiative flux.

2.2. *The case of a black surface*

In this case the kernel of equation (6) is

$$K(\tau,\tau') = \frac{1}{2}E_1(|\tau-\tau'|) + \frac{1}{2}\beta E_1(2a-\tau-\tau') + \gamma E_2(a-\tau)E_2(a-\tau'), \tag{13a}$$

where $E_n(x)$ is the exponential integral of order n, and we sēt $\beta_2 = \beta, \gamma_2 = \gamma$. The source term of equation (6) is given by

$$S(\tau) = 2\alpha_1 E_2(a+\tau) + [2\alpha_2 + 4\gamma\alpha_1 E_3(2a)] \times E_2(a-\tau) + 2\beta\alpha_1 E_2(3a-\tau). \tag{13b}$$

The matrix elements A_{mn} of the system of equations (11) can be expressed as

$$A_{mn} = \frac{[(2m+1)(2n+1)]^{1/2}}{2a} \times \left\{\frac{1}{2}C_{mn} + \frac{1}{2}\beta E_{mn} + \gamma D_m D_n\right\}, \tag{14a}$$

whereas for the known terms B_m we obtain

$$B_m = \left(\frac{2m+1}{2a}\right)^{1/2}\{[(-1)^m 2\alpha_1 + 2\alpha_2 + 4\gamma\alpha_1 E_3(2a)]D_m + 2\beta\alpha_1 F_m\}. \tag{14b}$$

The coefficients C_{mn}, E_{mn}, D_m and F_m, occurring in equations (14a) and (14b), are listed in Appendix II.

The solution $I_0(\tau)$ to equation (6) can thus be cast in the form

$$I_0(\tau) = c\sum_{n=0}^{\infty}\left(\frac{2n+1}{2a}\right)^{1/2}\xi_n\left\{\frac{1}{2}U_n(\tau) + \frac{1}{2}\beta U_n^*(\tau) + \gamma D_n E_2(a-\tau)\right\} + S(\tau), \tag{15}$$

the functions $U_n(\tau)$ and $U_n^*(\tau)$ being reported in Appendix III. Once $I_0(\tau)$ is known, it is introduced in equations (4) to yield the angular radiation intensity $(0 \leqslant \mu \leqslant 1)$

$$I(\tau,\mu) = \frac{c}{2\pi}\sum_{n=0}^{\infty}\left(\frac{2n+1}{2a}\right)^{1/2}\xi_n\left\{\frac{1}{2}W_n(\tau,\mu)\right\} + \frac{1}{2\pi}2\alpha_1\exp\left(-\frac{a+\tau}{\mu}\right) \tag{16a}$$

and

$$I(\tau,-\mu) = \frac{c}{2\pi}\sum_{n=0}^{\infty}\left(\frac{2n+1}{2a}\right)^{1/2}\xi_n\left\{\frac{1}{2}(-1)^n W_n(-\tau,\mu) + \left(\frac{1}{2}\beta W_n(a,\mu) + \gamma D_n\right)\exp\left(-\frac{a-\tau}{\mu}\right)\right\} + \frac{1}{2\pi}\left\{[2\alpha_2 + 4\gamma\alpha_1 E_3(2a)]\exp\left(-\frac{a-\tau}{\mu}\right) + 2\beta\alpha_1\exp\left(-\frac{3a-\tau}{\mu}\right)\right\}, \tag{16b}$$

where the function $W_n(\tau, \mu)$ is explicitly given in Appendix IV.

According to equation (8) we can finally evaluate the radiation flux vector intensity

$$q(\tau) = \left\{\frac{c}{2}\sum_{n=0}^{\infty}\left(\frac{2n+1}{2a}\right)^{1/2}\xi_n G_n(\tau) + 2\alpha_1 E_3(a+\tau)\right\} - \left\{\frac{c}{2}\sum_{n=0}^{\infty}\left(\frac{2n+1}{2a}\right)^{1/2}\xi_n[(-1)^n G_n(-\tau) + 2\gamma D_n E_3(a-\tau) + \beta H_n(\tau)] + [2\alpha_2 + 4\gamma\alpha_1 E_3(2a)]E_3(a-\tau) + 2\beta\alpha_1 E_3(3a-\tau)\right\}, \quad (17)$$

the functions $G_n(\tau)$ and $H_n(\tau)$ being given in Appendix V. It must be underlined that the coefficients C_{mn}, E_{mn}, D_m, F_m, and the functions $U_n(\tau)$, $U_n^*(\tau)$, $W_n(\tau, \mu)$, $G_n(\tau)$, $H_n(\tau)$, listed in the appendices, are all evaluable in terms of elementary operations and are analytically expressed in terms of special functions, like Gegenbauer polynomials, exponential integrals and incomplete gamma functions [12]. The spectral radius of the linear integral operator generated by the kernel $K(\tau, \tau')$, equation (13a), can be estimated as shown in [11]. The result of this estimation confirms the convergence of the iterative procedure in $L_2(-a, a)$ since

$$\rho \leqslant \operatorname*{ess.\,sup.}_{\tau\in(-a,a)} \int_{-a}^{a} |K(\tau,\tau')|\mathrm{d}\tau' \leqslant 1 - \frac{1-(\beta+\gamma)}{2}E_2(2a), \quad (18)$$

which is less than $1/c$ provided one of the two numbers c or $(\beta + \gamma)$ is less than 1, and the other is not greater than 1.

The case $\alpha_1 = 1$, $\alpha_2 = 0$ (i.e. emission only by the surface $\tau = -a$) has been considered in the numerical processing of the relevant formulae given above; results for all the significant physical quantities are given in Figs. 1–4. The total radiation intensity I_0 is, of course, extremely sensitive to the presence of a reflecting surface at the boundary, $\tau = a$, at least in the range of low and medium optical thickness, i.e. in those situations where enough energy is available for the reflection. The resulting effects on the I_0 distribution are obviously more relevant the higher albedo is.

Diffuse reflection is more effective than the specular kind in increasing the local values of I_0, as shown by the curves 7 and 5 of Fig. 1 which refer to the cases $\gamma = 1$, $\beta = 0$, and $\gamma = 0$, $\beta = 1$ respectively. This trend is also confirmed by a comparison of the curves 6 and 5 of the same figure which allows one to infer that when total reflection occurs at the wall, $\tau = a$, the presence of a

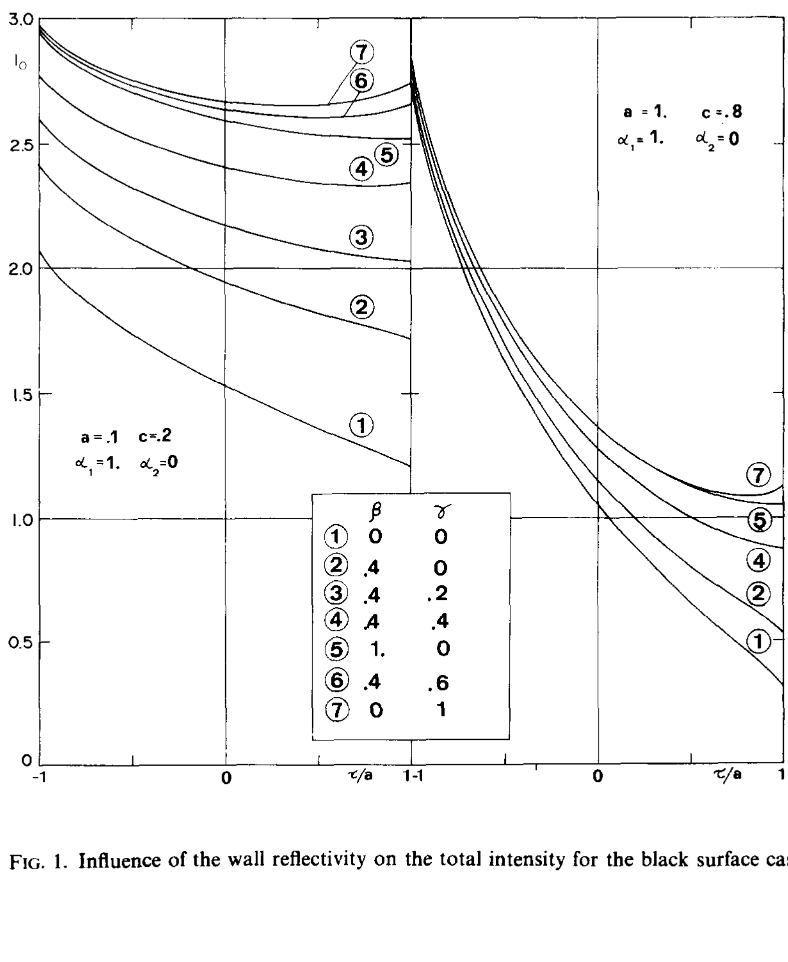

FIG. 1. Influence of the wall reflectivity on the total intensity for the black surface case.

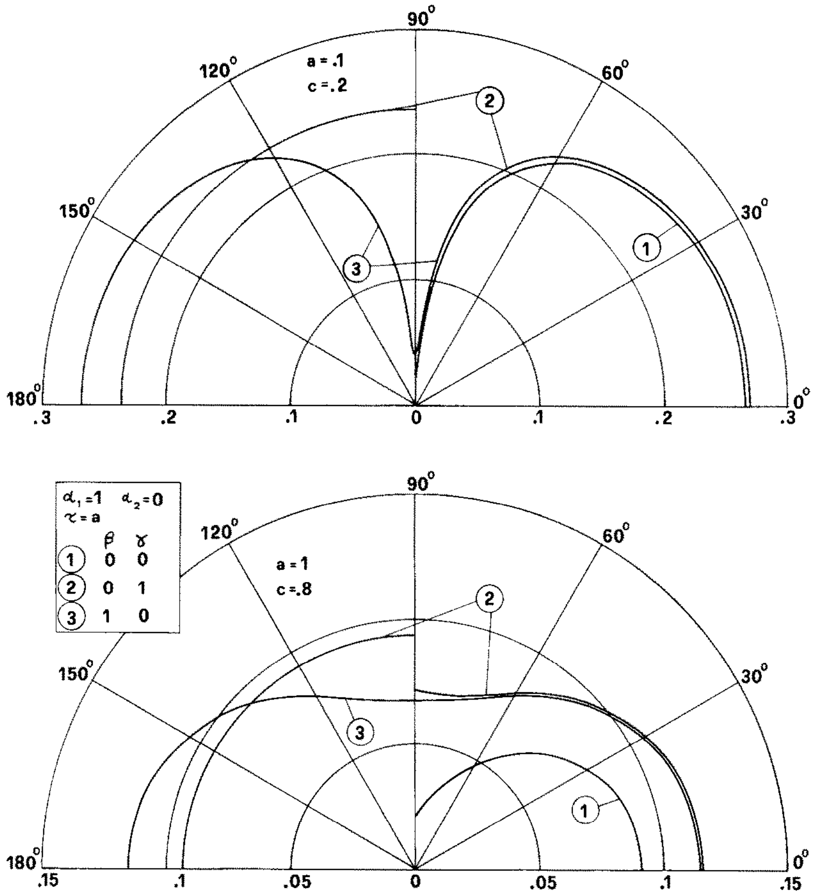

FIG. 2. Influence of the wall reflectivity on the angular intensity at the reflecting wall for the black surface case.

diffuse component of the wall reflectivity enhances the values of I_0. With regard to the angular intensity $I(\tau, \mu)$, results are given at the boundary, $\tau = a$, (Fig. 2) and at the mid-slab plane (Fig. 3) for different values of a and c.

For $\mu > 0$, the $I(\tau, \mu)$ curves are affected by the reflection at the boundary but are slightly sensitive to the reflection mode and are practically coincident when pure specular and diffuse total reflection are considered. For $\mu < 0$ ($\theta > 90°$), on the contrary, the $I(\tau, \mu)$ curves differ markedly depending on the reflection mode.

The trend exhibited by the $I(\tau, \mu)$ curves is relevant to explain the very slight dependence of the q^+ and q^- curves, and then of the q curve, on the reflection mode at the reflecting boundary. When total reflection is considered, q^+ curves are practically independent of the reflection mode since the corresponding $I(\tau, \mu)$ curves are practically coincident for $\mu > 0$, while for $\mu < 0$, higher values result for $I(\tau, \mu)$ in the diffuse reflection case than in the specular one, essentially for $\mu \to 0$, in such a way that the difference between the two cases is reduced when q^-, i.e. the first moment of $I(\tau, \mu)$, is evaluated.

2.3. *The case of zero specular reflectivity*

In this case the kernel of equation (6) is given by

$$K(\tau, \tau') = \frac{1}{2} E_1(|\tau - \tau'|) + \frac{1}{1 - 4\gamma_1\gamma_2 E_3^2(2a)} \times \{\gamma_1 E_2(a + \tau) E_2(a + \tau') + \gamma_2 E_2(a - \tau) E_2(a - \tau') + 2\gamma_1\gamma_2 E_3(2a)[E_2(a+\tau)E_2(a-\tau') + E_2(a - \tau)E_2(a + \tau')]\}, \quad (19a)$$

which is symmetric like the kernel of equation (13a) for the case of a black surface, whereas for the source term $S(\tau)$ we have

$$S(\tau) = \frac{1}{1 - 4\gamma_1\gamma_2 E_3^2(2a)} \{[2\alpha_1 + 4\gamma_1\alpha_2 E_3(2a)] \times E_2(a+\tau) + [2\alpha_2 + 4\gamma_2\alpha_1 E_3(2a)] E_2(a-\tau)\}. \quad (19b)$$

Following the scheme adopted for the case of a black

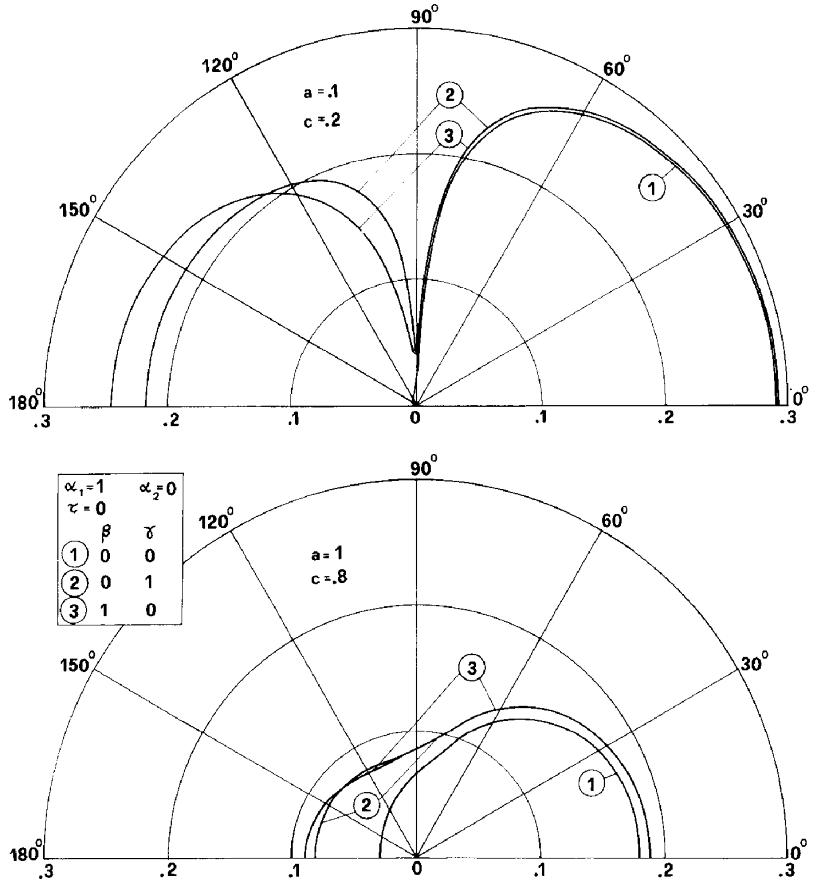

FIG. 3. Influence of the wall reflectivity on the angular intensity at the mid-slab plane for the black surface case.

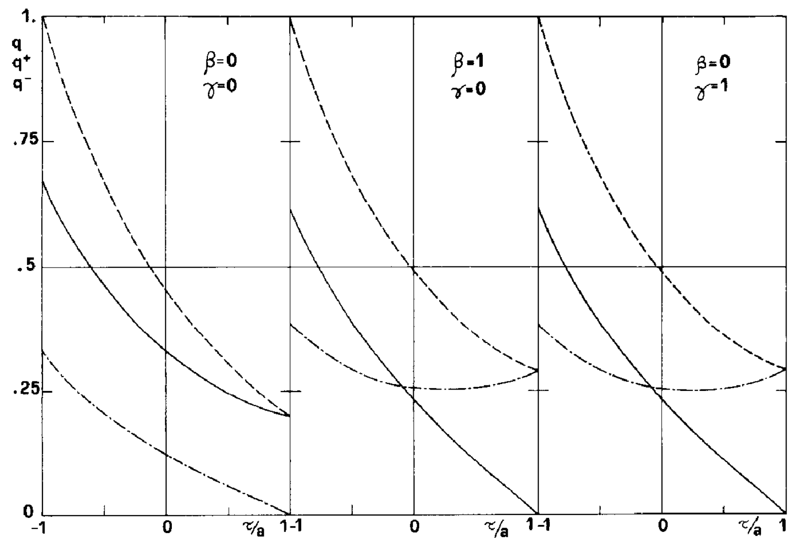

FIG. 4. Influence of the wall reflectivity on the forward flux q^+ (- - -), the backward flux q^- (- · -) and the net flux q (——) for the black surface case ($a = 1$, $c = 0.8$, $\alpha_1 = 1$, $\alpha_2 = 0$).

surface, the expressions below result for the general matrix element

$$A_{mn} = \frac{[(2m+1)(2n+1)]^{1/2}}{2a}\left\{\frac{1}{2}C_{mn} + \frac{(-1)^{m+n}\gamma_1[1+(-1)^n 2\gamma_2 E_3(2a)] + \gamma_2[1+(-1)^n 2\gamma_1 E_3(2a)]}{1-4\gamma_1\gamma_2 E_3^2(2a)} \times D_m D_n\right\} \quad (20a)$$

and the general known term of the system of equations (11)

$$B_m = \left[\frac{(2m+1)}{2a}\right]^{1/2}[1-4\gamma_1\gamma_2 E_3^2(2a)]^{-1} \times D_m\{(-1)^m[2\alpha_1 + 4\gamma_1\alpha_2 E_3(2a)] + [2\alpha_2 + 4\gamma_2\alpha_1 E_3(2a)]\}. \quad (20b)$$

For the total radiation intensity $I_0(\tau)$, from equation (6), we then get

$$I_0(\tau) = c[1-4\gamma_1\gamma_2 E_3^2(2a)]^{-1}\sum_{n=0}^{\prime}\left[\frac{(2n+1)}{2a}\right]^{1/2}\xi_n \times \left\{\frac{1}{2}U_n(\tau)+(-1)^n\gamma_1[1+(-1)^n 2\gamma_2 E_3(2a)] \times D_n E_2(a+\tau)+\gamma_2[1+(-1)^n 2\gamma_1 E_3(2a)] \times D_n E_2(a-\tau)\right\}+S(\tau). \quad (21)$$

The corresponding equations to (16a) and (16b) for $I(\tau,\mu)$ and $I(\tau,-\mu)$ are now ($0 \leqslant \mu \leqslant 1$)

$$I(\tau,\mu) = \frac{c}{2\pi}\sum_{n=0}^{\prime}\left(\frac{2n+1}{2a}\right)^{1/2}\xi_n\left\{\frac{1}{2}W_n(\tau,\mu) + \frac{(-1)^n\gamma_1[1+(-1)^n 2\gamma_2 E_3(2a)]}{1-4\gamma_1\gamma_2 E_3^2(2a)} \times D_n \exp\left(-\frac{a+\tau}{\mu}\right)\right\} + \frac{1}{2\pi} \times \frac{2\alpha_1+4\gamma_1\alpha_2 E_3(2a)}{1-4\gamma_1\gamma_2 E_3^2(2a)}\exp\left(-\frac{a+\tau}{\mu}\right) \quad (22a)$$

and

$$I(\tau,-\mu) = \frac{c}{2\pi}\sum_{n=0}^{\prime}\left(\frac{2n+1}{2a}\right)^{1/2}\xi_n \times \left\{\frac{1}{2}(-1)^n W_n(-\tau,\mu) + \frac{\gamma_2[1+(-1)^n 2\gamma_1 E_3(2a)]}{1-4\gamma_1\gamma_2 E_3^2(2a)} \times D_n \exp\left(-\frac{a-\tau}{\mu}\right)\right\} + \frac{1}{2\pi}\,\frac{2\alpha_2+4\gamma_2\alpha_1 E_3(2a)}{1-4\gamma_1\gamma_2 E_3^2(2a)}\exp\left(-\frac{a-\tau}{\mu}\right). \quad (22b)$$

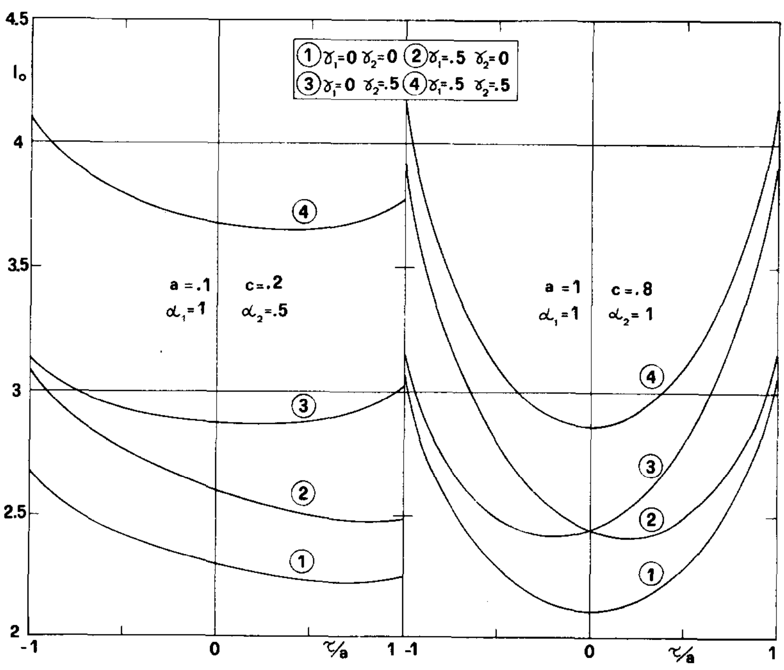

FIG. 5. Influence of the walls' reflectivity on the total intensity for the case of zero specular reflectivity.

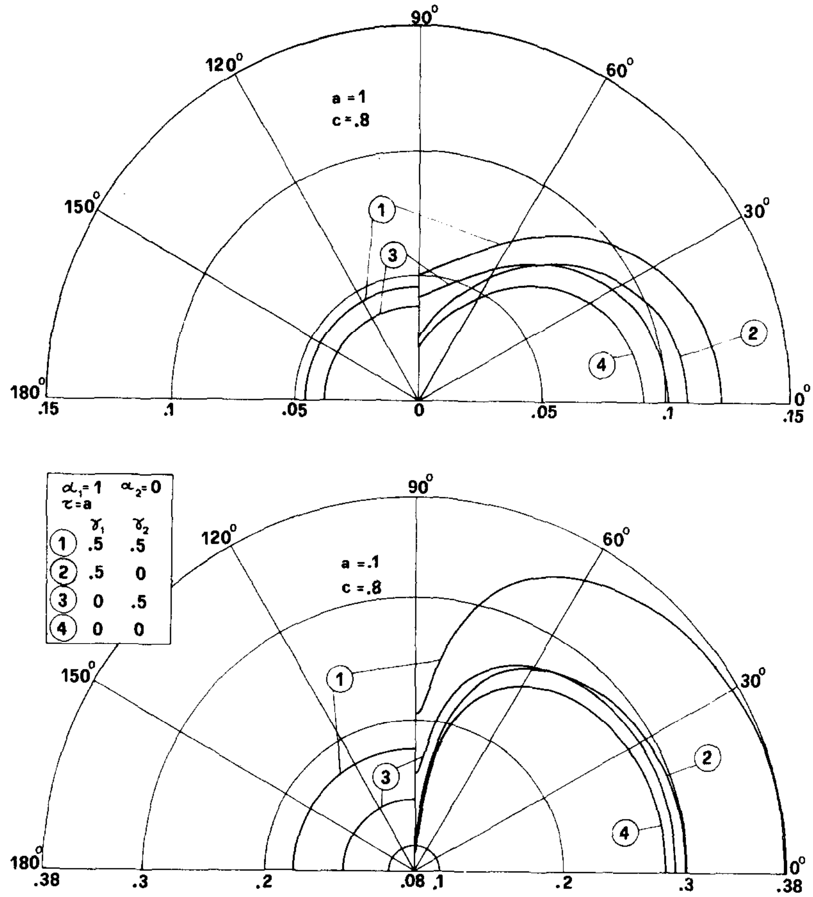

Fig. 6. Influence of the walls' reflectivity on the angular intensity at the wall $\tau = a$ for the case of zero specular reflectivity.

For the net radiative flux it follows that

$$q(\tau) = \frac{c}{2}\sum_{n=0}^{\prime}\left(\frac{2n+1}{2a}\right)^{1/2}\xi_n\{G_n(\tau)-(-1)^n G_n(-\tau)$$
$$+ 2D_n[1-4\gamma_1\gamma_2E_3^2(2a)]^{-1}[(-1)^n\gamma_1$$
$$\times (1+(-1)^n 2\gamma_2E_3(2a))E_3(a+\tau)$$
$$-\gamma_2(1+(-1)^n 2\gamma_1E_3(2a))E_3(a-\tau)]\}$$
$$+ [1-4\gamma_1\gamma_2E_3^2(2a)]^{-1}[(2\alpha_1+4\gamma_1\alpha_2E_3(2a))$$
$$\times E_3(a+\tau)-(2\alpha_2+4\gamma_2\alpha_1E_3(2a))E_3(a-\tau)]\}. \tag{23}$$

The coefficients C_{mn}, D_m, and the functions $U_n(\tau)$, $W_n(\tau,\mu)$, $G_n(\tau)$ in equations (20)–(23), are the same as the ones already defined for the case of a black surface, and can be found in the appendices.

For the spectral radius ρ of the linear integral operator generated by the kernel $K(\tau,\tau')$, equation (19a), we have

$$\rho = \operatorname{ess.sup.}_{\tau\in(-a,a)}\int_{-a}^{a}|K(\tau,\tau')|\,d\tau' \leqslant 1 - \frac{1}{2}$$
$$\times \frac{(1-\gamma_1)(1+2\gamma_2E_3(2a))+(1-\gamma_2)(1+2\gamma_1E_3(2a))}{1-4\gamma_1\gamma_2E_3^2(2a)}$$
$$\times E_2(2a) < 1. \tag{24}$$

For ρ to be less than 1, it is sufficient that only one of the three numbers c, γ_1, γ_2 be less than 1, the other two both being allowed to assume the value one.

Numerical results have been obtained for the cases $\alpha_1 = 1, \alpha_2 = 0, 0.5, 1$ and for various combinations of γ_1 and γ_2.

Since only diffuse reflection has been considered at the walls, the analysis of the results here is more straight forward than in the black surface case: the influence of the walls reflectivity on the significant physical quantities can be easily grasped from Figs. 5–9.

REFERENCES

1. V. C. Boffi, F. Santarelli and C. Stramigioli, Integral transform method in radiative transfer, *J. Quant. Spectrosc. Radiat. Transfer* **18**, 189 (1977).
2. V. C. Boffi, V. G. Molinari and G. Spiga, Anisotropy of scattering and fission in neutron transport theory, *Nucl. Sci. Engng* **64**, 823 (1977).

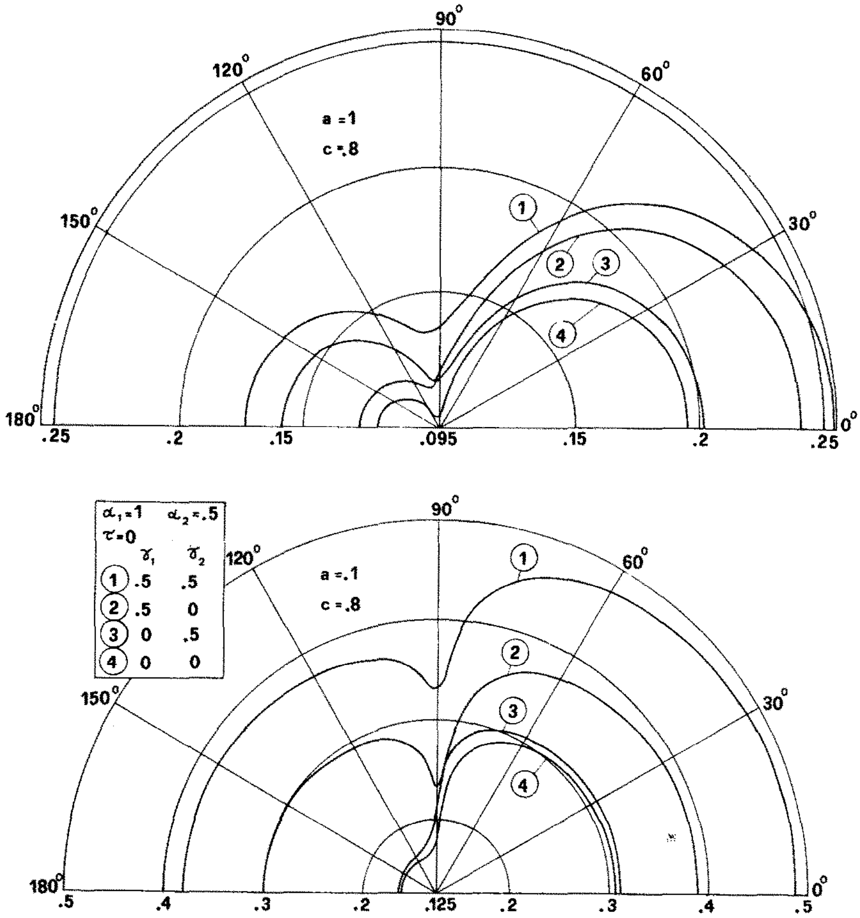

FIG. 7. Influence of the walls' reflectivity on the angular intensity at the mid-slab plane for the case of zero specular reflectivity.

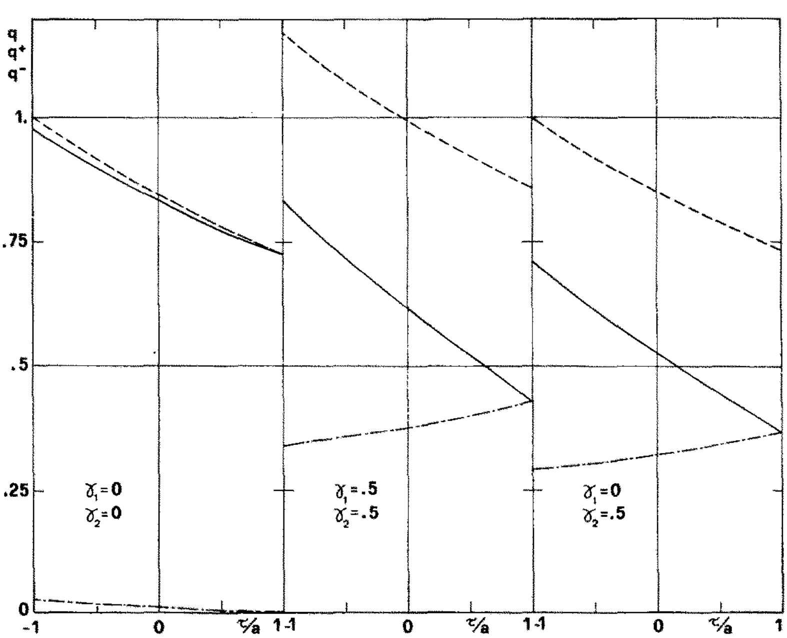

FIG. 8. Influence of the walls' reflectivity on the forward flux q^+ (– – –), the backward flux q^- (– · –) and the net flux q (——) for the case of zero specular reflectivity (a = 0.1, c = 0.2, α_1 = 1, α_2 = 0).

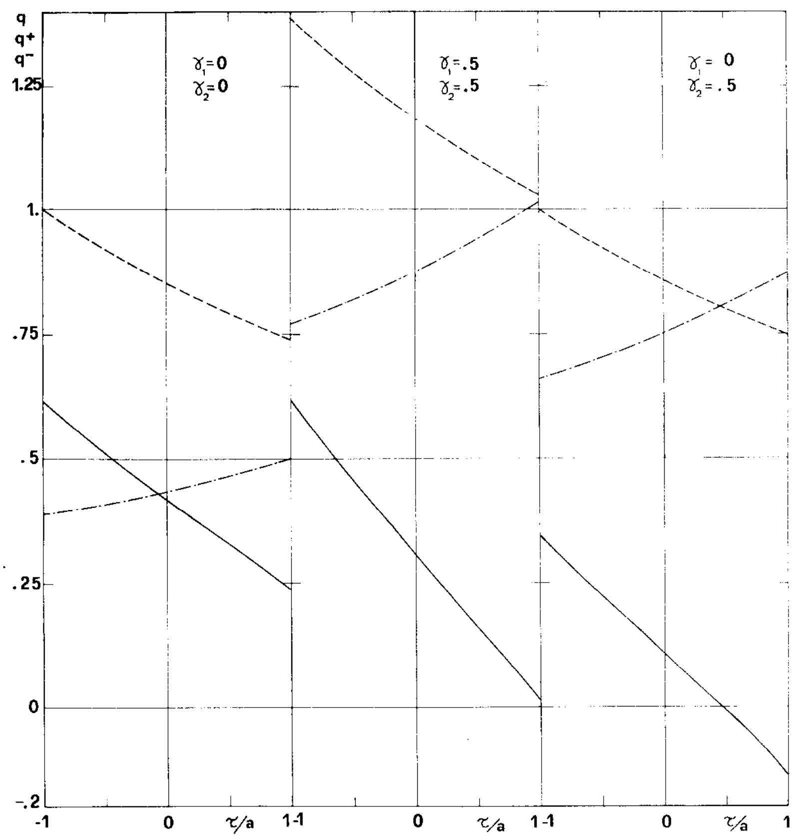

FIG. 9. Influence of the walls' reflectivity on the forward flux q^+ (– – –), the backward flux q^- (– · –) and the net flux q (——) for the case of zero specular reflectivity ($a = 0.1$, $c = 0.2$, $\alpha_1 = 1$, $\alpha_2 = 0.5$).

3. M. N. Ozisik and C. E. Siewert, On the normal-mode expansion technique for radiative transfer in a scattering, absorbing and emitting slab with specularly reflecting boundaries, *Int. J. Heat Mass Transfer* **12**, 611 (1969).
4. C. C. Lii and M. N. Ozisik, Hemispherical reflectivity and transmissivity of an absorbing isotropically scattering slab with a reflecting boundary, *Int. J. Heat Mass Transfer* **16**, 685 (1973).
5. H. L. Beach, M. N. Ozisik and C. E. Siewert, Radiative transfer in linearly anisotropic-scattering conservative and non conservative slabs with reflective boundaries, *Int. J. Heat Mass Transfer* **14**, 1551 (1971).
6. V. C. Boffi and G. Spiga, Integral theory of radiative heat transfer with anisotropic scattering and general boundary conditions, *J. Math. Phys.* **18**, 2448 (1977).
7. E. Cupini, A. de Matteis, F. Premuda and T. Trombetti, Numerical applications of a new approach to the solution of the neutron transport equation, C.N.E.N. Report RT/FI (69) 45, Rome (1969).
8. L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces*. Pergamon, Oxford (1964).
9. A. C. Zaanen, *Linear Analysis*. North Holland, Amsterdam (1964).
10. V. C. Boffi, *Fisica de Reattore Nucleare. La Teoria del Transporto del Neutroni*. Vol. I and II. Patron, Bologna (1974).
11. V. C. Boffi and G. Spiga, Linear integral transformations generated by the three-dimensional neutron transport kernel, *J. Math. Phys.* **14**, 1913 (1973).
12. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1970).

APPENDIX I

$$H_i(\tau,\mu) = 2\pi\left\{\beta_i \exp\left(-\frac{a-(-1)^i\tau}{\mu}\right) + \gamma_i\mu\int_0^1 g_i(\mu')\exp\left(-\frac{a-(-1)^i\tau}{\mu'}\right)d\mu'\right\} \qquad i = 1,2,$$

$$S_0(\tau) = 2\pi\int_0^1 \frac{1}{\mu}d\mu\left\{\int_{-a}^{\tau}\exp\left(-\frac{\tau-\tau'}{\mu}\right)Q(\tau',\mu)d\tau' + \int_{\tau}^{a}\exp\left(\frac{\tau-\tau'}{\mu}\right)Q(\tau',-\mu)d\tau'\right\}$$
$$+ \alpha_1\int_0^1 \exp\left(-\frac{a+\tau}{\mu}\right)h_1(\mu)d\mu + \alpha_2\int_0^1 \exp\left(-\frac{a-\tau}{\mu}\right)h_2(\mu)d\mu,$$

$$\tilde{K}_i(\mu,\tau) = (1-\beta_1\beta_2\exp(-4a/\mu))^{-1}\sum_{j=1}^{2}\left[\Delta_{i,3-j}^{(j)}(\mu)\tilde{H}_j(\mu,\tau)\right.$$
$$\left. + \Delta_{i,j}^{(3-j)}(\mu)\tilde{G}_j(\tau)\gamma_j g_j(\mu)\exp(-2a/\mu)\right] \qquad i = 1,2,$$

$$\Delta_{i,l}^{(j)}(\mu) = 1 - \delta_{il}(1 - \beta_j \exp(-2a/\mu)) \qquad i,j,l = 1,2,$$

$$\tilde{H}_i(\mu,\tau) = \frac{1}{\mu}\exp\left(-\frac{a-(-1)^i\tau}{\mu}\right) \qquad i = 1,2,$$

$$\tilde{G}_i(\tau) = \frac{1}{D}\left\{\left[1 - \int_0^1 \mu' \frac{\beta_i\gamma_{i'}g_{i'}(\mu')\exp(-4a/\mu')}{1-\beta_1\beta_2\exp(-4a/\mu')}\mathrm{d}\mu'\right]\right.$$

$$\times \int_0^1 \mu \frac{\sum_{j=1}^{2}\Delta_{i,3-j}^{(j)}(\mu)\tilde{H}_j(\mu,\tau)}{1-\beta_1\beta_2\exp(-4a/\mu)}\mathrm{d}\mu + \left(\int_0^1 \mu' \frac{\gamma_{i'}g_{i'}(\mu')\exp(-2a/\mu')}{1-\beta_1\beta_2\exp(-4a/\mu')}\mathrm{d}\mu'\right)$$

$$\left.\times \int_0^1 \mu \frac{\sum_{j=1}^{2}\Delta_{i,j}^{(j)}(\mu)\tilde{H}_j(\mu,\tau)}{1-\beta_1\beta_2\exp(-4a/\mu)}\mathrm{d}\mu\right\} \qquad \begin{matrix} i = 1,2 \\ i' \neq i \end{matrix},$$

$$D = \begin{vmatrix} 1 - \int_0^1 \mu \frac{\beta_2\gamma_1 g_1(\mu)\exp(-4a/\mu)}{1-\beta_1\beta_2\exp(-4a/\mu)}\mathrm{d}\mu & -\int_0^1 \mu \frac{\gamma_2 g_2(\mu)\exp(-2a/\mu)}{1-\beta_1\beta_2\exp(-4a/\mu)}\mathrm{d}\mu \\ -\int_0^1 \mu \frac{\gamma_1 g_1(\mu)\exp(-2a/\mu)}{1-\beta_1\beta_2\exp(-4a/\mu)}\mathrm{d}\mu & 1 - \int_0^1 \mu \frac{\beta_1\gamma_2 g_2(\mu)\exp(-4a/\mu)}{1-\beta_1\beta_2\exp(-4a/\mu)}\mathrm{d}\mu \end{vmatrix},$$

$$j_i(\mu) = (1-\beta_1\beta_2\exp(-4a/\mu))^{-1}\sum_{j=1}^{2}[\Delta_{i,3-j}^{(j)}(\mu)\tilde{F}_j(\mu)$$

$$+ \Delta_{i,j}^{(3-j)}(\mu)\delta_j\gamma_j g_j(\mu)\exp(-2a/\mu)] \qquad i = 1,2,$$

$$\tilde{F}_i(\mu) = \frac{1}{\mu}\int_{-a}^{a}\exp\left(-\frac{a-(-1)^i\tau}{\mu}\right)Q(\tau,(-1)^i\mu)\mathrm{d}\tau$$

$$+ \frac{\alpha_{i'}}{2\pi}h_{i'}(\mu)\exp(-2a/\mu) \qquad \begin{matrix} i = 1,2, \\ i' \neq i, \end{matrix}$$

$$\delta_i = \frac{1}{D}\int_0^1 \mu\,\mathrm{d}\mu\left\{\left[1 - \int_0^1 \mu' \frac{\beta_i\gamma_{i'}g_{i'}(\mu')\exp(-4a/\mu')}{1-\beta_1\beta_2\exp(-4a/\mu')}\mathrm{d}\mu'\right]\right.$$

$$\times \frac{\sum_{j=1}^{2}\Delta_{i,3-j}^{(j)}(\mu)\tilde{F}_j(\mu)}{1-\beta_1\beta_2\exp(-4a/\mu)} + \left(\int_0^1 \mu' \frac{\gamma_{i'}g_{i'}(\mu')\exp(-2a/\mu')}{1-\beta_1\beta_2\exp(-4a/\mu')}\mathrm{d}\mu'\right)$$

$$\left.\times \frac{\sum_{j=1}^{2}\Delta_{i,j}^{(j)}(\mu)\tilde{F}_j(\mu)}{1-\beta_1\beta_2\exp(-4a/\mu)}\right\} \qquad \begin{matrix} i = 1,2, \\ i' \neq i. \end{matrix}$$

APPENDIX II

$$C_{m,n} = \int_{-a}^{a} P_m(x/a)\mathrm{d}x \int_{-a}^{a} E_1(|x-y|)P_n(y/a)\mathrm{d}y$$

$$= [1 + (-1)^{m+n}]2a^2 \sum_{v=0}^{m+n+1} \frac{\beta_v^{mn}}{v!}\{E_1(2a) + v!V_v(2a)\},$$

$$V_v(z) = \frac{1}{v!z^{v+1}}\int_0^z x^v \exp(-x)\mathrm{d}x = \frac{1}{z^{v+1}}\left[1 - \exp(-z)\sum_{k=0}^{v}\frac{z^k}{K!}\right],$$

$$\beta_v^{mn} = \begin{cases} \dfrac{2\delta mn}{2m+1} & v = 0, \\ -1 & v = 1, \\ \dfrac{2(-1)^v}{(v-1)!(v+1)}\prod_{K=1}^{v-1}(m+n+1+v-2K)(|m-n|+v-2K) & v = 2,\ldots,m+n+1, \end{cases}$$

$$D_m = \int_{-a}^{a} E_2(a-x)P_m(x/a)\mathrm{d}x$$

$$= \sum_{K=0}^{m} \frac{(-1)^K(2K-1)!!(m+K)!}{a^K(2K)!(m-K)!}\left\{\frac{(2a)^{K+1}E_2(2a)}{(K+1)!} + \frac{(2a)^{K+2}E_1(2a)}{(K+2)!}\right.$$

$$\left. + \frac{(2a)^{K+2}}{K+2}V_{K+1}(2a)\right\},$$

$$E_{mn} = \int_{-a}^{a} P_m(x/a)\mathrm{d}x \int_{-a}^{a} E_1(2a-x-y)P_n(y/a)\mathrm{d}y,$$

$$= 2a^2 \sum_{\nu=0}^{m+n+1} (\nu+1)\beta_\nu^{mn} \left\{ (-1)^{n+\nu} \sum_{K=0}^{\nu} \frac{(-1)^K 2^{K+1}}{(K+1)!(\nu-K)!} [E_1(4a) + K!V_K(4a)] \right.$$

$$\left. + [(-1)^m - (-1)^{n+\nu}] \sum_{K=0}^{\nu} \frac{(-1)^K}{(K+1)!(\nu-K)!} [E_1(2a) + K!V_K(2a)] \right\} \quad \text{for } m \geq n$$

$(E_{mn} = E_{nm}$ when $m < n)$,

$$F_m = \int_{-a}^{a} E_2(3a-x)P_m(x/a)\mathrm{d}x$$

$$= 2a \sum_{K=0}^{m} \frac{2^K(2K-1)!!(m+K)!}{(2K)!(m-K)!} \sum_{j=0}^{K} \frac{(-1)^j}{(j+1)!(K-j)!} \left\{ 2^{j+1} \left[E^2(4a) \right. \right.$$

$$\left. + \frac{4a}{j+2} E_1(4a) + \frac{4a}{j+2}(j+1)!V_{j+1}(4a) \right] - \left[E_2(2a) \right.$$

$$\left. \left. + \frac{2a}{j+2} E_1(2a) + \frac{2a}{j+2}(j+1)!V_{j+1}(2a) \right] \right\}.$$

APPENDIX III

$$U_n(x) = \int_{-a}^{a} E_1(|x-y|)P_n(y/a)\mathrm{d}y$$

$$= \sum_{K=0}^{n} \frac{(-1)^K(2K-1)!!}{a^K(K+1)!}(a+x)^{K+1}C_{n-K}^{(K+1/2)}(x/a)\{E_1(a+x)$$

$$+ K!V_K(a+x)\} + \sum_{K=0}^{n} \frac{(2K-1)!!}{a^K(K+1)!}(a-x)^{K+1}C_{n-K}^{(K+1/2)}(x/a)\{E_1(a-x)$$

$$+ K!V_K(a-x)\},$$

$C_{n-K}^{(K+1/2)}(x/a)$ is Gegenbauer ultraspherical polynomial;

$$U_n^*(x) = \int_{-a}^{a} E_1(2a-x-y)P_n(y/a)\mathrm{d}y$$

$$= \sum_{K=0}^{n} (-1)^{n-K} 2^K (2K-1)!! C_{n-K}^{(K+1/2)}(x/a) \sum_{j=0}^{K} \frac{(-1)^j}{(2a)^j(j+1)!(K-j)!}$$

$$\times \{(3a-x)^{j+1}[E_1(3a-x) + j!V_j(3a-x)] - (a-x)^{j+1}[E_1(a-x)$$

$$+ j!V_j(a-x)]\}.$$

APPENDIX IV

$$W_n(x,\mu) = \frac{1}{\mu}\int_{-a}^{x} \exp\left(-\frac{x-y}{\mu}\right)P_n(y/a)\mathrm{d}y$$

$$= \frac{1}{\mu}\sum_{K=0}^{n} \frac{(-1)^K(2K-1)!!}{a^K}(a+x)^{K+1}C_{n-K}^{(K+1/2)}(x/a)V_K\left(\frac{a+x}{\mu}\right).$$

APPENDIX V

$$G_n(x) = \int_{-a}^{x} E_2(x-y)P_n(y/a)\mathrm{d}y$$

$$= \sum_{K=0}^{n} \frac{(-1)^K(2K-1)!!}{a^K}(a+x)^{K+1}C_{n-K}^{(K+1/2)}(x/a)\left\{\frac{1}{(K+1)!}E_2(a+x)\right.$$

$$\left. + \frac{a+x}{(K+2)!}E_1(a+x) + \frac{a+x}{K+2}V_{K+1}(a+x)\right\},$$

$$H_n(x) = \int_{-a}^{a} E_2(2a-x-y)P_n(y/a)\mathrm{d}y$$

$$= 2a(-1)^n \sum_{K=0}^{n} (-1)^K 2^K (2K-1)!! C_{n-K}^{(K+1/2)}(x/a) \sum_{j=0}^{K} \frac{(-1)^j}{(j+1)!(K-j)!}$$

$$\times \left\{ \left(\frac{3a-x}{2a}\right)^{j+1} \left[E_2(3a-x) + \frac{3a-x}{j+2}E_1(3a-x) + \frac{3a-x}{j+2} \right. \right.$$

$$\left. \times (j+1)!V_{j+1}(3a-x) \right] - \left(\frac{a-x}{2a}\right)^{j+1} \left[E_2(a-x) + \frac{a-x}{j+2}E_1(a-x) \right.$$

$$\left. \left. + \frac{a-x}{j+2}(j+1)!V_{j+1}(a-x) \right] \right\}.$$

TRANSFERT RADIATIF DANS UNE PLAQUE ABSORBANTE ET DIFFUSANTE LIMITEE PAR DES SURFACES EMETTRICES ET REFLECHISSANTES

Résumé—On présente une théorie intégrale exacte du transfert radiatif plan avec une diffusion isotrope et des conditions aux limites générales. La solution analytique du problème est numériquement développée pour deux spécialisations différentes de l'émissivité et de la réflectivité des surfaces frontières. Des résultats sont donnés pour les intensités totales et angulaires et pour le flux net de rayonnement.

STRAHLUNGSAUSTAUSCH AN EINER ABSORBIERENDEN UND STREUENDEN PLATTE, DIE VON EMITTIERENDEN UND REFLEKTIERENDEN OBERFLÄCHEN BEGRENZT WIRD

Zusammenfassung—In dieser Arbeit wird eine exakte integrale Theorie des Strahlungs austausches einer ebenen Fläche mit isotroper Streuung und allgemeinen Randbedingungen angegeben. Die analytische Lösung des Problems wird numerisch für zwei verschiedene Spezialfälle der Emissions- und der Reflexionseigenschaften der umgebenden Oberflächen durchgeführt. Ergebnisse werden sowohl für die Gesamtstrahlung als auch für die Richtungsverteilung und den Netto-Energiestrom der Strahlung angegeben.

ЛУЧИСТЫЙ ПЕРЕНОС В ПЛОСКОМ ПОГЛОЩАЮЩЕМ И РАССЕИВАЮЩЕМ СЛОЕ, ОГРАНИЧЕННОМ ИЗЛУЧАЮЩЕЙ И ОТРАЖАЮЩЕЙ ПОВЕРХНОСТЯМИ

Аннотация — Излагается интегральная теория процесса двумерного лучистого переноса при наличии изотропного рассеивания и общих граничных условий. Проведена численная обработка аналитического решения для двух различных случаев: излучающей и отражающей ограничивающих поверхностей. Приводятся результаты расчётов суммарных и угловых интенсивностей излучения, а также результирующего лучистого потока.